12. Monday, July 15. **First countable spaces**

Theorem 1: Let X and Y be Top spaces.

1) If $A \subseteq X$ is closed, then it is sequentially closed.

2) If $f: X \to Y$ is continuous at $x \in X$, then it is sequentially continuous at x.

Proof: HW!

Theorem 2: If X is first countable, the converses of (1) and (2) hold.

Proof: (1) Assume A is sequentially closed. WTS A is closed, or equivalently $X \setminus A$ is open.
Let $x \in X \setminus A$. Take a nbhd basis $\mathcal{B}_x = \{B_n \mid n \in \mathbb{N}\}$ as in Lemma 3.

Claim: $\exists\, n \in \mathbb{N}$ such that $B_n \subseteq X \setminus A$.

Otherwise, $\forall n \in \mathbb{N}$ we have $B_n \cap A \neq \emptyset$. Take $x_n \in B_n \cap A$, and consider the sequence $(x_n)_{n \in \mathbb{N}}$.

Since $x_n \in B_n\ \forall n$, by lemma 3 (2), we know $x_n \to x$.

Since $x_n \in A\ \forall n$ and A is seq. closed we have $x \in A$, contradicts $x \in X \setminus A$.

Hence $X \setminus A$ is a neighborhood of each of its points, hence it is open.

2) Assume f seq. continuous at x. Let V nbhd of $f(x)$. WTS $\exists$ nbhd U of x such that $U \subseteq f^{-1}(V)$

Let $\mathbb{B}_x = \{B_n \mid n \in \mathbb{N}\}$ as in lemma 3.

Claim: $\exists$ $n \in \mathbb{N}$ such that $B_n \subseteq f^{-1}(V)$.

Assume for a contradiction that's not the case. Then for each $n \in \mathbb{N}$ there is some $z_n \in B_n \setminus f^{-1}(V)$. Consider the sequence $(z_n)_{n \in \mathbb{N}}$.

Since $z_n \in B_n$ $\forall n$, we know that $z_n \to x$ by lemma 11.3.

Since f is seq. continuous at x, we deduce that $f(z_n) \to f(x)$ in Y.

But, since V is a nbhd of $f(x)$, there must be some $N \in \mathbb{N}$ such that $f(z_N) \in V$

This contradicts $z_n \notin f^{-1}(V)$ $\forall n \in \mathbb{N}$.

So the claim must be true and it follows that f is continuous at x $\square$

Quotient spaces (§22)

Recall: A relation on a set X is a subset $R \subseteq X \times X$. We denote $x \sim y$ instead of $(x,y) \in R$.
We say R is an eq. relation if it is

- Reflexive: $x \sim x \quad \forall x$
- Symmetric $x \sim y \Rightarrow y \sim x$
- Transitive $x \sim y$ and $y \sim z \Rightarrow x \sim z$.

Construction: Assume $\sim$ is an eq. relation on X.

1) For each $x \in X$, The equivalence class of x is $[x] = \{ y \in X \mid x \sim y \}$

2) The quotient set of X is $X/_{\sim} = \{ [x] \mid x \in X \}$

3) The quotient map is the function

$$X \longrightarrow X/_{\sim} \quad \text{where} \quad \pi(x) = [x] \quad \forall x \in X.$$

Which is surjective by definition of $X/_{\sim}$

In practice: We use the quotient set to identify certain points in X.

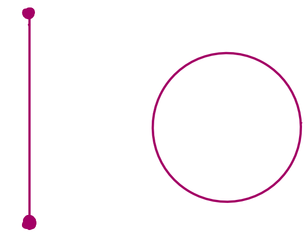
Example: Let $X = [0,1]$. Define an equivalence relation by

$$x \sim y \iff x = y \text{ or } x, y \in \{0,1\}$$

In other words, internal points are only related to themselves, and the two extreme points are related. The classes for this relation are

$$[0,1]/_{\sim} = \{\{x\} \mid x \in (0,1)\} \cup \{\{0,1\}\}$$

So we "paste" 0 with 1 and leave all other points intact



Seems like $[0,1]/_{\sim} = S^1$, but they are not the same set.

Theorem (Functions out of quotients)

Let $\sim$ eq. relation on X. Let $f: X \longrightarrow Y$ be a function.

1) Assume that, for all $x, y \in X$, if $x \sim y$ then $f(x) = f(y)$. Then there exists a unique function $\bar{f}: X/_{\sim} \longrightarrow Y$ such that $\bar{f}([x]) = f(x)$ $\forall x \in X$.

In other words, $\bar{f} \circ \pi = f$

2) In this case $\bar{f}(X/\sim) = f(X)$.

3) Assume that, for all $x, y \in X$, we have $x \sim y \iff f(x) = f(y)$. Then $\bar{f}$ is injective.

Definition: If $f : X \to Y$ is such that $x \sim y \Rightarrow f(x) = f(y)$, we say that f descends to the quotient (or also that f factors through the quotient)

Example: In the example $[0,1]/\sim$ from before, to define a function $[0,1]/\sim \longrightarrow S^1$ just need to define a function $f : [0,1] \longrightarrow S^1$ such that $f(0) = f(1)$

For example $f(t) = (\cos(2\pi t), \sin(2\pi t))$ satisfies $f(0) = f(1)$ and hence induces

$\bar{f} : [0,1]/\sim \longrightarrow S^1$ where $\bar{f}([t]) = f(t) = (\cos(2\pi t), \sin(2\pi t))$.

Furthermore, $\bar{f}$ is surjective because so is f.

And for $x, y \in [0,1]$, we know that $f(x) = f(y) \iff x = y$, so $\bar{f}$ is injective.

In other words, $\bar{f} : [0,1]/\sim \longrightarrow S^1$ is bijective.

Next Time: If X is a topological space, induce topology on $X/\sim$.