Previously: A tree is a connected graph with no cycles. For any tree T we know $|E(T)| = |V(T)| - 1$.

Theorem (4.7). *If* G *is a connected graph, then* $|E(G)| \geqslant |V(G)| - 1$.

Theorem (4.8). *If a graph* G *satisfies any two of the properties*

(1) G *is connected,* *(2)* G *contains no cycles.* *(3)* $|E(G)| = |V(G)| - 1$*,*

then G *is a tree.*

Theorem. *Let* T *be a tree on* k *vertices. If* G *is a graph with* $\delta(G) \geqslant k-1$*, then* T *is isomorphic to some subgraph of* G.

Example. *Can you identify all trees on 4 vertices as subgraphs of the Peterson graph?*

Recall that a subgraph H of a graph G is spanning if $V(H) = V(G)$.

Definition. *A* ***spanning tree*** *of a connected graph* G *is a spanning subgraph* $H \subseteq G$ *that is also a tree.*

Theorem. *Every connected graph contains a spanning tree.*

Exercise 1 (Corollary 4.6)**.** *If a forest has exactly n vertices and k components, then it has $n - k$ edges.*

Exercise 2. *Show that every tree is bipartite.*

Exercise 3. *Prove that a graph G is a tree if and only if G contains no cycle but $G + uv$ does contain a cycle for each pair of non-adjacent vertices u, v in G.*

Exercise 4. *Let T be a tree. For each $i \geqslant 1$, let n_i denote the number of vertices of degree i. Show that*

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \dots$$