Last weeks:

- Vertex coloring and edge coloring
- Chromatic numbers
- Critical graphs
- Factorization

This week:

- Planar graphs
- Planarity and chromatic numbers

Motivation:

- There are three houses.
- Want to connect each house to three utilities: water, gas, electric.
- Utility lines can't cross each other.

Is that possible?

Definition. • *A graph is called **planar** if it can be drawn in the plane so that no two edges cross each other.*

• *A **plane** graph is a graph that is drawn in the plane so that no two edges cross each other.*

Note: A plane graph is just a valid drawing of a planar graph.

Example (A planar graph can have more than one drawing.)**.**

Note: The motivating problem translates to:

Facts and Definitions Let G be a plane graph. Then

(1) G breaks the plane into a collection of connected regions. These regions are called the **faces of** G.

(2) The set of all faces is denoted $F(G)$.

(3) There is exactly one unbounded face, which we refer to as the **exterior face**.

(4) All other faces are called **interior faces.**

(5) Trees are plane graphs which exactly one face (the exterior face).

Example.

More facts and Definitions Let G be a plane graph. Then

(6) A vertex/edge is **incident** to a face if that vertex/edge lives on the boundary of that face.

(7) A face could be incident to many vertices, and a vertex could be incident to many faces.

(8) An edge can be incident to at most two faces.

(9) An edge e is incident to exactly one face of G if and only if e is a bridge.

Example.

Theorem (Euler's formula, 9.1)**.** *If* G *is a connected plane graph, then*

$$|V(G)| + |F(G)| - |E(G)| = 2.$$

Theorem (Euler's formula)**.** *If* G *is any plane graph, then*

$$|V(G)| + |F(G)| - |E(G)| = 1 + \#\{\textit{components of } G\}.$$

Corollary. *If* G *is a planar graph, then any two drawings of* G *on the plane have the same number of faces.*

Definition. *Let* G *be a plane graph and let* $\mathsf{f} \in \mathsf{F}(\mathsf{G})$*. The* ***length of*** f *is*

$$\textit{len}(\mathsf{f}) = \#\{\textit{edges incident to } \mathsf{f}\} + \#\{\textit{bridges incident to } \mathsf{f}\}.$$

Intuition: Walk around the boundary of the face, counting each edge you encounter.
Bridges are counted twice, because you need to cross it twice to walk around the boundary.

Warning: This depends on the drawing, not just the graph.

Example.

Headshaking Lemma. If G is a plane graph, then

$$\sum_{f \in F(G)} \text{len}(f) = 2|E(G)|.$$

Theorem (9.2)**.** *Let* G *be a connected planar graph on* $n \geqslant 3$ *vertices. Then*

$$|E(G)| \leqslant 3n - 6.$$

Corollary (9.4). *The clique* K_5 *is not a planar graph.*

Question: Is $K_{3,3}$ planar?

Theorem (9.2)**.** *Let* G *be a connected planar graph on* $n \geqslant 3$ *vertices. If* G *is triangle-free, then*

$$|E(G)| \leqslant 2n - 4.$$

Corollary (9.5). *The complete bipartite graph* $K_{3,3}$ *is not a planar graph.*