Name: ____.

Student number: ______________________________________.

Instructions

- Turn off all the electronic devices.
- This is a closed book exam.
- You are allowed a notesheet. You can use any fact in the notesheet, but you must reference which fact you are using.
- Unless otherwise stated, you must justify your answers.
- If you have a question, raise your hand and I will come to you.
- You have 50 minutes to complete the exam.
- Good luck!

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

1. (10 points) Let X be a topological space and A, B subsets of X. Determine which statements below are true, and which are false. (If your answer is True, you must prove it; if your answer is False, you must provide a counter example.)

   (a) $(A \cup B)^\circ = A^\circ \cup B^\circ$

False. Let $X = \mathbb{R}$ with standard topology

Let $A = (0,1]$ and $B = (1,2)$

Then $A^\circ = (0,1)$, $B^\circ = (1,2)$, so $A^\circ \cup B^\circ = (0,1) \cup (1,2)$

On the other hand, $A \cup B = (0,2)$, so

$(A \cup B)^\circ = (0,2)$.

Thus $A^\circ \cup B^\circ \neq (A \cup B)^\circ$

   (b) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

False. Let X, A and B as before.

Then $\overline{A} = [0,1]$ and $\overline{B} = [1,2]$

Hence $\overline{A} \cap \overline{B} = \{1\}$

On the other hand, $A \cap B = \emptyset$, so

$\overline{A \cap B} = \emptyset$.

Thus $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

2. (10 points) Let (X, d_X) and (Y, d_Y) be metric spaces. And consider the set $Z = X \times Y$. Define a function $d_Z \colon Z \times Z \to \mathbb{R}$ by

$$d_Z\big((x,y),\ (x',y')\big) = d_X(x,x') + d_Y(y,y').$$

(a) Prove that d_Z is a metric on Z.

1) Symmetric:

$d((x,y),(x',y')) = d_X(x,x') + d_Y(y,y') = d_X(x',x) + d_Y(y',y) = d_Z((x',y'),(x,y))$

2) Non-degeneracy:

$d_Z((x,y),(x',y')) = d_X(x,x') + d_Y(y,y') \geq 0$ with equality if and only if both $d_X(x,x') = 0$ and $d_Y(y,y') = 0$, which happens if and only if $x = x'$ and $y = y'$, or in other words $(x,y) = (x',y')$.

3) Δ-inequality:

$$\begin{aligned} d_Z((x,y),(x'',y'')) &= d_X(x,x'') + d_Y(y,y'') \\ &\leq d_X(x,x') + d_X(x',x'') + d_Y(y,y') + d_Y(y',y'') \\ &= d_Z((x,y),(x',y')) + d_Z((x',y'),(x'',y'')) \end{aligned}$$

(b) Let $\mathcal{T}$ be the topology on Z induced by the metric d_Z, and $\mathcal{T}'$ be the product topology on $Z = X \times Y$. Show that $\mathcal{T} = \mathcal{T}'$.

<u>Recall</u>: $\mathcal{T}$ has basis $\mathbb{B}_d = \{ B((x,y), r) \mid (x,y) \in Z,\ r > 0 \}$

$\mathcal{T}'$ has basis $\mathbb{B}_{Prod} = \{ U \times V \subseteq X \times Y \mid U \text{ open in } X,\ V \text{ open in } Y$

$[\mathcal{T}' \subseteq \mathcal{T}]$ It's enough to show that the basis $\mathbb{B}_{Prod}$ for $\mathcal{T}'$ satisfies $\mathbb{B}_{Prod} \subseteq \mathcal{T}$. Fix $U \times V \in \mathbb{B}_{Prod}$. Let $(x,y) \in U \times V$ fix $\varepsilon > 0$ such that $B_X(x, \varepsilon) \subseteq U$ and $B_Y(y, \varepsilon) \subseteq V$, Thus $B_X(x, \varepsilon) \times B_Y(y, \varepsilon) \subseteq U \times V$

<u>Claim</u>: $B_Z((x,y), \varepsilon) \subseteq B_X(x, \varepsilon) \times B_Y(y, \varepsilon)$

Indeed, if $d_Z((x,y),(x',y')) < \varepsilon$, then $d_X(x,x') \leq d_Z((x,y),(x',y')) < \varepsilon$ and also $d_Y(y,y') \leq d_Z((x,y),(x',y')) < \varepsilon$

Thus $(x',y') \in B_X(x, \varepsilon) \times B(y, \varepsilon)$. This proves the claim.

Then, for each $(x,y) \in U \times V$ there exists $B_Z((x,y), r) \in \mathbb{B}_d$ with $(x,y) \in B_Z((x,y), r) \subseteq U \times V$. By the local description of $\mathcal{T} = \mathcal{T}(\mathbb{B}_d)$, we have $U \times V \in \mathcal{T}$.

$[\mathcal{T} \subseteq \mathcal{T}']$. Let $W \in \mathcal{T}$ and $(x,y) \in W$. Then $\exists\, \varepsilon > 0$ such that $B_Z((x,y), \varepsilon) \subseteq W$.

<u>Claim</u>: $B_X(x, \frac{\varepsilon}{2}) \times B_Y(y, \frac{\varepsilon}{2}) \subseteq B_Z((x,y), \varepsilon) \subseteq W$

Indeed, if $d_X(x,x') < \frac{\varepsilon}{2}$ and $d_Y(y,y') < \frac{\varepsilon}{2}$, Then

$d_Z((x,y),(x',y')) = d_X(x,x') + d_Y(y,y') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

So $(x',y') \in B_Z((x,y), \varepsilon)$, which proves the claim.

As before, using the local description of $\mathcal{T}' = \mathcal{T}(\mathbb{B}_{Prod})$, This says that $W \in \mathcal{T}'$, as desired.

3. (10 points) Let X be a topological space. Let $x \in X$ and let A be a subset of X.

(a) Prove that $x \in \overline{A}$ if and only if every open set U containing x intersects A

Recall $\overline{A} = \bigcap_{\substack{C \text{ closed} \\ A \subseteq C}} C$.

[$\Rightarrow$] Assume $x \in \overline{A}$. Let U open containing x, and assume for a contradiction $U \cap A = \emptyset$. Then $C^* = X \setminus U$ is closed and contains A (because $A \cap U = \emptyset$) Since $x \in \overline{A} = \bigcap_{\substack{C \text{ closed} \\ A \subset C}} C$, we have in particular $x \in C^* = X \setminus U$, which contradicts $x \in U$. Therefore $U \cap A \neq \emptyset$.

[$\Leftarrow$] Assume $x \in X$ is such that every open U containing x satisfies $U \cap A \neq \emptyset$ Let C closed with $A \subseteq C$. Then $U = X \setminus C$ is open, and since $A \subseteq C$, we have $U \cap A = \emptyset$. Therefore U cannot contain x, which means $x \in C$. Hence $x \in \overline{A}$.

(b) We say x is a *cluster point* of A if, for every open set U containing x, we have $A \cap (U \setminus \{x\}) \neq \emptyset$. The set of all cluster points of A is denoted by A'. Prove that $\overline{A} = A \cup A'$.

[$\subseteq$] Let $x \in \overline{A}$. WTS either $x \in A$ or $x \in A'$. If $x \in A$ we are done, so assume $x \notin A$ and let's prove $x \in A'$. Let U open containing x. By part (a), since $x \in \overline{A}$, we have $U \cap A \neq \emptyset$, but since $x \notin A$, this implies $(U \setminus \{x\}) \cap A \neq \emptyset$.

[$\supseteq$] Need to show $A \subseteq \overline{A}$ and $A' \subseteq \overline{A}$. We have $A \subseteq \overline{A}$ by definition of $\overline{A} = \bigcap_{\substack{C \text{ closed} \\ A \subseteq C}} C$. So, let's show $A' \subseteq \overline{A}$

If $x \in A'$ and U is open with $x \in U$, we have $(U \setminus \{x\}) \cap A \neq \emptyset$. Hence also $U \cap A \neq \emptyset$. By part (a) we have $x \in \overline{A}$.

4. (10 points) Let $\omega \notin \mathbb{R}$ and define $X = \mathbb{R} \cup \{\omega\}$. For each $x \in X$ and $r > 0$, define

$$A(x,r) = \begin{cases} \{y \in \mathbb{R} \mid |y-x| < r\}, & \text{if } x \in \mathbb{R} \\ \{y \in \mathbb{R} \mid |y| > r\} \cup \{\omega\}, & \text{if } x = \omega. \end{cases}$$

(a) Show that $\mathcal{A} = \{A(x,r) \mid x \in X,\ r > 0\}$ is a basis for a topology on X.

We need to verify:

1) $X = \bigcup_{A \in \mathcal{A}} A$.

2) Given $A_1, A_2 \in \mathcal{A}$ and $x \in A_1 \cap A_2$, there exists $A_x \in \mathcal{A}$ with $x \in A_x \subseteq A_1 \cap A_2$.

1) is clear, because for any $x \in X$ we have $x \in A(x,1)$

2) Let $A(x,r)$ and $A(x',r')$ in $\mathcal{A}$. Let $y \in A(x,r) \cap A(x',r')$.

•) In case $y = \omega$, since $\omega \in A(x,r)$ we have $x = \omega$. Similarly, $x' = \omega$.
Thus, if we let $r'' = \max\{r, r'\}$, then $\omega \in A(\omega, r'') \subseteq A(\omega, r) \cap A(\omega, r')$

•) In the other case, we have $y \in \mathbb{R}$. Since $A(x,r) \setminus \{\omega\}$ and $A(x',r') \setminus \{\omega\}$ are open sets in $(\mathbb{R}, \mathcal{T}_{std})$, also is their intersection, which contains y.
Hence, by definition of $\mathcal{T}_{std}$, there is some $r'' > 0$ such that
$(y - r'', y + r'') \subseteq (A(x,r) \setminus \{\omega\}) \cap (A(x',r') \setminus \{\omega\})$
Hence $A(y,r) = (y - r'', y + r'') \subseteq A(x,r) \cap A(x',r')$, as desired.

Therefore $\mathcal{A}$ defines a basis for a topology on X

(b) Provide X with the topology generated by $\mathcal{A}$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that f extends to a continuous function $g\colon X \to \mathbb{R}$ if and only if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist in $\mathbb{R}$ and are equal.

Recall: for a function $f: \mathbb{R} \to \mathbb{R}$, we define:

1) $\lim_{x\to+\infty} f(x) = \ell_1 \in \mathbb{R} \iff \forall \varepsilon>0\ \exists R>0$ such that if $x>R$ then $|f(x)-\ell_1|<\varepsilon$.

2) $\lim_{x\to-\infty} f(x) = \ell_2 \in \mathbb{R} \iff \forall \varepsilon>0\ \exists R>0$ such that if $x<-R$ then $|f(x)-\ell_2|<\varepsilon$

($\Rightarrow$) Assume $f:\mathbb{R}\to\mathbb{R}$ is a continuous function that can be extended continuously $X \to \mathbb{R}$. In other words, $\exists\, g: X \to \mathbb{R}$ continuous such that $g|_{\mathbb{R}} = f$. Let $\ell = g(\omega) \in \mathbb{R}$. We prove that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty}$ both exist and are equal to ℓ.

Let $\varepsilon>0$. Since $(\ell-\varepsilon, \ell+\varepsilon) \subseteq \mathbb{R}$ is open and $g: X\to\mathbb{R}$ is continuous with $g(\omega)=\ell$, there exists a neighborhood $A(\omega, R)$ of ω in X such that $g(A(\omega,R)) \subseteq (\ell-\varepsilon, \ell+\varepsilon)$

Let $x \in \mathbb{R}$ with $x>R$. Then $x \in A(\omega,R)$, so $f(x)=g(x) \in g(A(\omega,R)) \subseteq (\ell-\varepsilon, \ell+\varepsilon)$.

In other words, if $x>R$ then $|f(x)-\ell|<\varepsilon$.

This works $\forall \varepsilon>0$, so $\lim_{x\to\infty} f(x) = \ell$

Similarly, given $x \in \mathbb{R}$ with $x<-R$, we have $x \in A(\omega,R)$, so again $f(x) \in (\ell-\varepsilon, \ell+\varepsilon)$ as before. In other words, $x<-R \Rightarrow |f(x)-\ell|<\varepsilon$

This works $\forall\varepsilon>0$, so $\lim_{x\to-\infty} f(x) = \ell$.

(b) Provide X with the topology generated by $\mathcal{A}$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that f extends to a continuous function $g\colon X \to \mathbb{R}$ if and only if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist in $\mathbb{R}$ and are equal.

$\Leftarrow$) Assume $f:\mathbb{R}\to\mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \ell \in \mathbb{R}$

Thus $\forall \varepsilon>0$ $\exists R>0$ such that if $|x|>R$ then $|f(x)-\ell|<\varepsilon$.

Define $g: X \to \mathbb{R}$ by $g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \\ \ell & \text{if } x = \omega \end{cases}$. We prove g continuous

Let $U \subseteq \mathbb{R}$ open. WTS $g^{-1}(U)$ open in X. We consider two cases

Case 1: $\ell \notin U$. Then $g^{-1}(U) = f^{-1}(U)$. Since f is continuous, $f^{-1}(U) \subseteq \mathbb{R}$ is open. Hence $\forall x \in f^{-1}(U)$, $\exists$ $r>0$ such that $B(x,r) \subseteq f^{-1}(U)$. But this just says that $\forall x \in g^{-1}(U)$, $\exists$ $r>0$ such that $A(x,r) \subseteq g^{-1}(U)$.
Hence $g^{-1}(U)$ is open in X by the local description of the topology $\mathcal{T}(\mathcal{A})$.

Case 2: $\ell \notin U$. Then $g^{-1}(U) = f^{-1}(U) \cup \{\omega\}$. We show $g^{-1}(U) \subseteq X$ is open using the local description of $\mathcal{T}(\mathcal{A})$. Let $x \in g^{-1}(U)$. If $x \in f^{-1}(U)$, as in the previous case, $\exists$ $r>0$ such that $A(x,r) \subseteq g^{-1}(U)$. Assume thus $x=\omega$. Since $\ell \in U$ and $U \subseteq \mathbb{R}$ is open, $\exists$ $\varepsilon>0$ such that $(\ell-\varepsilon, \ell+\varepsilon) \subseteq U$. Since $\lim_{x\to\infty} f(x) = \ell = \lim_{x\to-\infty} f(x)$, for this $\varepsilon>0$, $\exists$ $R>0$ such that $|x|>R \Rightarrow |f(x)-\ell|<\varepsilon$
In other words, $|x|>R \Rightarrow g(x) = f(x) \in (\ell-\varepsilon, \ell+\varepsilon) \subseteq U$
Since also $g(\omega) \in (\ell-\varepsilon, \ell+\varepsilon)$, this means that $A(\omega, R) \subseteq g^{-1}(U)$

In any case, $g^{-1}(U)$ is open. Hence g is continuous.