Last week:

- Degree sequences.
- Graphical sequences, Havel–Hakimi theorem, Erdös–Gallai (one direction).
- Adjacency matrix (how to store a graph).
- Graph isomorphisms and automorphisms.

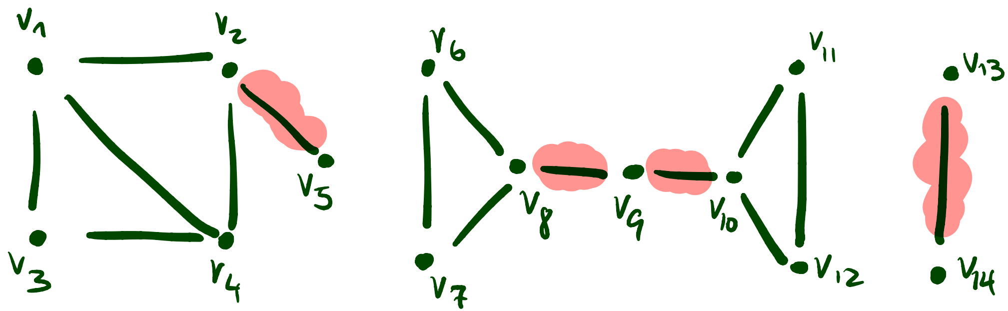
This week:

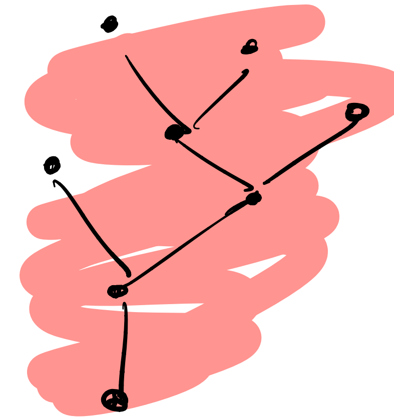
- Trees.
- More trees.
- Spanning trees.

Definition. *An edge* e *of a graph* G *is a* ***bridge*** *if* G − e *has more components than* G.

- In case G is connected, e is a bridge if and only if G − e is disconnected.

Example. *Identify edges in the following graph.*





Exercise. (a) *If* e = uv *is a bridge in* G, *then* u, v *lie in different components of* G − e.

Let G_1 denote the component of G containing u,v.

u,v belong to different components of $G_1 - e$.

If there is a u-v path in $G_1 - e$ . . . Any x,y in $G_1 - e$ are connected.

(b) *If* e = uv *is a bridge in* G, *then there is a unique* u–v *path in* G.

→ If another path exists, use a).

Theorem (4.1). *An edge e of a graph G is a bridge if and only if e lies on no cycle of G.* Assume $e = uv$

$\Leftarrow$) Suppose uv lies on no cycle.

Claim: There are no u-v paths in $G-e$. Assume $P = (u = v_0, \ldots, v_n = v)$ is a u–v path in $G-e$

Then $(u = v_0, v_1, \ldots, v_n = v, u)$ is a cycle on G.

Thus u, v lie in different components of $G-e$.

Since any path in $G-e$ is a path in G, we have more components in $G-e$ than in G.

$\Rightarrow$) Assume $e = uv$ is a bridge. This means: $\exists\, x, y \in V(G)$ for which There is a path in G, but no path on $G-e$.

Thus any x–y path on G uses e. Wlog such a path uses u first and then v.

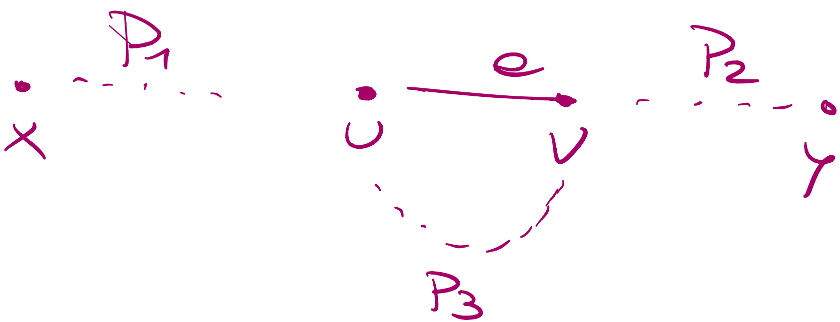
$$P = (x = w_0, w_1, \ldots, w_k = u, w_{k+1} = v, w_{k+2}, \ldots, w_m = y).$$

P_1 e P_2

Assume e lies on a cycle $C = (\underbrace{u = v_0, v_1, \dots, v_n = v}_{P_3}, u)$

Then $P_1 P_3 P_2$ is an x-y walk on $G \setminus e$

$\Rightarrow$ There is an x-y path on $G \setminus e$. ↯

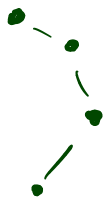


<u>Task</u>: Use exercise a) to assume $x = u$, $y = v$.

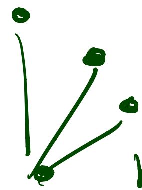
Definition. *A* ***tree*** *is a connected graph with no cycles.*

leaf: $v \in V$ / $\deg v = 1$.

Example (All trees on 4 vertices)**.**



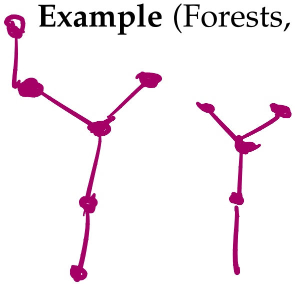
P_3, Path

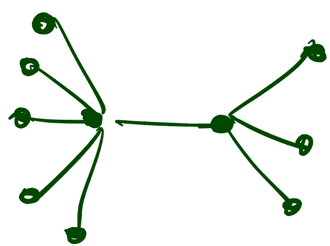


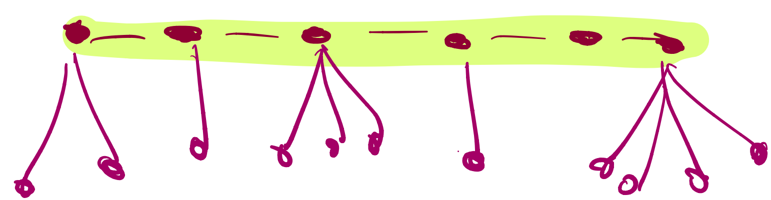
$K_{1,3}$ star $\subseteq$ complete bipartite

- A **forest** is a graph with no cycles.
- A tree is a star if it has exactly one vertex that is not a leaf.
- A tree is a **double star** if it has exactly two vertices that are not leaves.
- A tree on at least 3 vertices is a **caterpillar** if removing all leaves results in a path. That path is the **spine**.

Example (Forests, (double) stars, caterpillars)**.**







Theorem (Not in the book). *Let* G *be a graph. If there are* $x, y \in V(G)$ *connected by at least two paths, then* G *contains a cycle.*

Proof. Consider two different x–y paths $(x = u_0, u_1, \ldots, u_k = y)$ and $(x = v_0, v_1, \ldots, v_\ell = y)$, say with $k \leqslant \ell$.

Step 1: Let i denote the largest index for which $u_j = v_j$ for all $j \in \{0, \ldots, i\}$. Then $i < k$.

If $i = k$ $\rightsquigarrow$ $k = \ell$ Then the paths are the same.

$\rightsquigarrow$ $k < \ell$ $y = u_k = v_k$, $v_k = v_\ell$ with $k \neq \ell$, contrad.

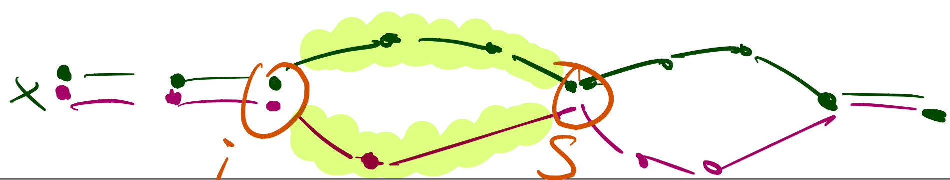
Step 2: There is a smallest $s \in \{i+1, \ldots, k\}$ for which $u_s \in \{v_{i+1}, \ldots, v_\ell\}$.

There is such s (namely $s = k$) ✓

Step 3: Let $t \in \{i+1, \ldots, \ell\}$ such that $u_s = v_t$. Then $s \neq i+1$ or $t \neq i+1$.

If $s = i+1 = t$ $u_{i+1} = u_s = v_t = v_{i+1}$ Contradicts 1.

Step 4: $(v_i = u_i, u_{i+1}, \ldots, u_s = v_t, v_{t-1}, \ldots v_{i+1})$ is a cycle in G. □



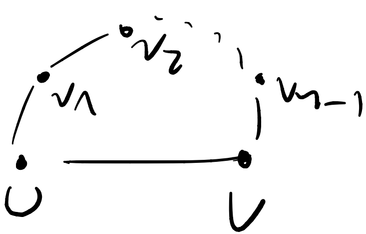
Theorem (4.2). *A graph G is a tree if and only if every two vertices of G are connected by a unique path.*

$\Rightarrow$) Assume G is a tree. Take $u, v \in V(G)$. Since G is connected, it contains a u-v path. If we have two u-v paths, then we can build a cycle (using previous Thm) on G.

$\Leftarrow$) WTS $\longrightarrow$ G connected ✓ (can connect any two vertices)
$\longrightarrow$ G contains no cycles.

If $C = (u = v_0 = v_1 \ldots, v_n = v, u)$ is a cycle

Then $(u = v_0, \ldots, v_n = v)$ different u-v paths.
(u, v)



Theorem (4.3). *Every nontrivial tree has at least two leaves.* $\to \deg v = 1$

Note: If T is a tree on n vertices and v is a leaf, then T − v is a tree with n − 1 vertices.

→ $T-v$ has $n-1$ vertices ✓

→ $T-v$ is connected (because $\deg v = 1$)

→ $T-v$ contains no cycle ✓

Proof: take a longest path $P = (u = v_0, v_1, \ldots, v_n = v)$

Claim: u and v are leaves. WTS $\deg u = 1$

If w is a neighbor of u

- w can't be outside of P (could build longer path $(w, u, v_1, \ldots, v_n = v)$
- w can't be any v_i with $i > 1$ because $(u = v_0, v_1, \ldots, w, u)$ is a cycle on G

Thus $w = v_1$, so $\deg u = 1$.

∎

Theorem (4.4). *If* T *is a tree, then* $|E(T)| = |V(T)| - 1$. *(A tree on* n *vertices has exactly* $n-1$ *edges.)*

Proof: By induction on $n = |V(T)|$ The case $n=1$ only contains the tree • which has 0 edges

Assume true for trees on $n-1$ vertices.

Take T tree on n vertices. Take a leaf $v \in T$ (4.3)

Then $T \setminus v$ is a tree on $n-1$ vertices

$$\underbrace{|E(T\setminus v)|}_{|E(T)|-\cancel{1}} = \underbrace{|V(T-v)|}_{|V(T)|-\cancel{1}} - 1 \quad (IH)$$

$$|E(T)| = |V(T)| - 1 \quad ✓$$

$\boxed{}$

Corollary (4.6). *If a forest has exactly n vertices and k components, then it has $n - k$ edges.*

Exercise. *Show that every tree is bipartite.*

Exercise. *Prove that a graph G is a tree if and only if G contains no cycle but $G + uv$ does contain a cycle for each pair of non-adjacent vertices u, v in G.*

Exercise. *Let T be a tree. For each $i \geqslant 1$, let n_i denote the number of vertices of degree i. Show that*

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \dots$$