Last week:

- Vertex coloring
- Chromatic numbers
- Critical graphs
- Chromatic numbers and trees

This week:

- Edge coloring

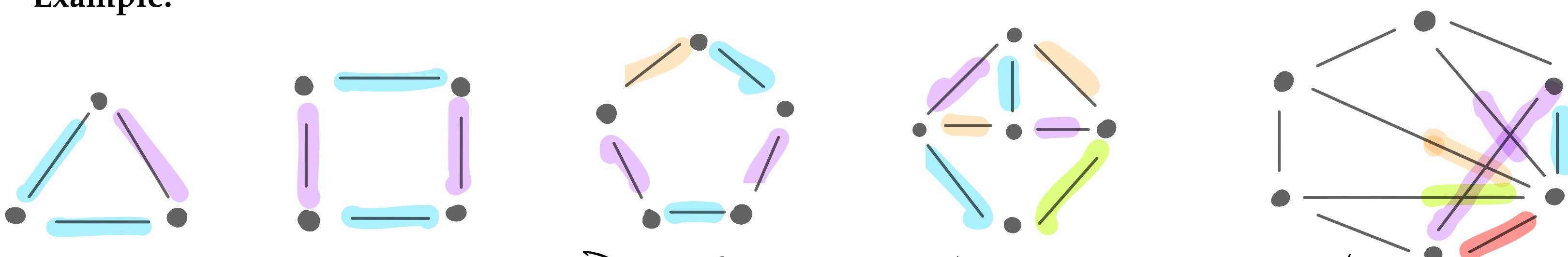
- Factorization
- Edge coloring

Definition. *Let* G *be a graph.*

- *An* ***edge-coloring*** *of* G *is a function* $f\colon E(G) \to C$ *where* C *is a non-empty set.*
- *Such edge-coloring is* ***proper*** *if* $f(e) \neq f(s)$ *whenever* $e, s \in E(G)$ *share a common vertex.*
- *For an integer* $t \geqslant 1$, *a* t***-edge-coloring*** *of* G *is a proper coloring* $f\colon E(G) \to C$ *such that* $|C| \leqslant t$.
- *The* ***edge-chromatic number*** $\chi'(G)$ *is the smallest integer* t *such that* G *has a proper* t*-edge-coloring.*

a.k.a chromatic index.

Example.



edge coloring not proper

Proper $\chi'(C_4)=2$

Proper $\chi'(C_5)=3$

$\chi'=4$

$\chi'=5$

$$\chi'(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases}$$

Quick observations:

(a) If G is an n-vertex graph, then $\chi'(G) \leqslant \binom{n}{2}$.

$|E(G)| \leq \binom{n}{2}$, just use a color for each edge.

(b) $\chi'(G) = 0$ if and only if G has no edges.

$\chi'(G) = 0 \Rightarrow$ no edges

no edges $\Rightarrow \chi'(G) = 0$

(c) If H is a subgraph of G, then $\chi'(H) \leqslant \chi'(G)$.

Any proper edge-coloring for G restricts to a proper edge coloring for H.

Let $f: E(G) \longrightarrow C = \{c_1, \dots, c_t\}$ proper coloring.

$f^{-1}(c_1) = \{e \in E(G) : f(e) = c_1\}$ is a matching

Proposition. *Proper edge-colorings of* G *using* t*-colors correspond to partitions of* E(G) *into* t *matchings. In particular,* $\chi'(G)$ *is the smallest integer for which we can partition* E(G) *into* t *matchings.*

Proof: Let $f: E(G) \to C = \{c_1, \dots, c_t\}$ proper edge coloring.

Let $M_i = f^{-1}(c_i) = \{e \in E(G) : f(e) = c_i\}$

Then $E(G) = M_1 \sqcup M_2 \sqcup \dots \sqcup M_t$

and each M_i is a matching in G: if we have $e, s \in M_i$

Then $f(e) = f(s)$, hence e, s are vertex-disjoint.

Conversely: if $E(G) = M_1 \sqcup \dots \sqcup M_t$, all M_i's matchings.

Define $f: E(G) \to \{1, \dots, t\}$ by $f(e) = i$ if $e \in M_i$

This is a proper coloring, because M_i's are matchings. $\blacksquare$

Theorem. *For any graph* G *we have*

$$\chi'(G) \geqslant \Delta(G), \quad \text{and} \quad \chi'(G) \geqslant \frac{|E(G)|}{\alpha'(G)}.$$

$\alpha'(G)$ → Edge-independence (size of largest matching)

Proof: $\boxed{\chi'(G) \geq \Delta(G)}$ Let $v \in V(G)$ such that $\deg v = \Delta(G)$. Then we have $\Delta(G)$ edges using v. Those $\Delta(G)$ have a common vertex, so any proper edge coloring needs at least $\Delta(G)$ colors.

$\left|\chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}\right.$: By previous prop, we have

$E(G) = M_1 \cup M_2 \cup \dots \cup M_{\chi'(G)}$, each M_i is a matching.

Then $|M_i| \leq \alpha'(G)$. Hence

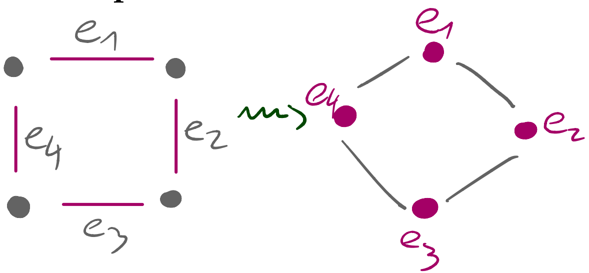
$$|E(G)| = \sum_{i=1}^{\chi'(G)} |M_i| \leq \sum_{i=1}^{\chi'(G)} \alpha'(G) = \alpha'(G)\,\chi'(G). \text{ i.e.: } \chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}$$

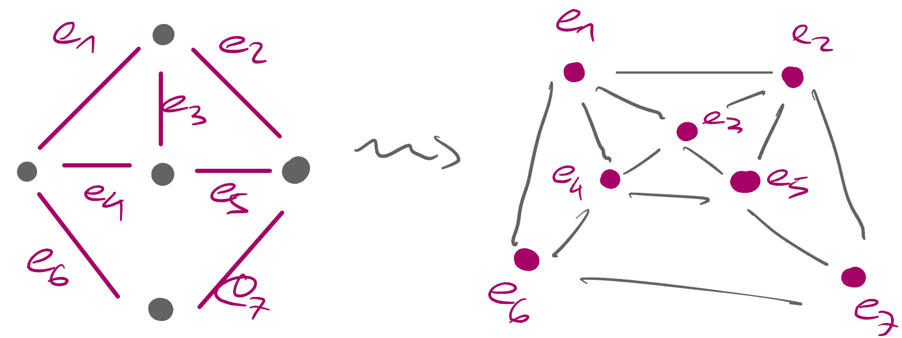
Next: Upper bounds $\chi'(G) \leq d(G)+1 \leq \Delta(G)+1$ ☒

Definition. *For a graph* $G = (V, E)$*, the* ***line graph*** $L(G)$ *has vertex set* E *and edges* $\{es : e \neq s \in E, e \cap s \neq \varnothing\}$

In other words, for each edge of G we have a vertex in $L(G)$, and for two edges in G with a common vertex, we have an edge in $L(G)$ between the corresponding vertices.

Example.





Exercise. (a) *Show that* $L(C_n) \simeq C_n$ *for any* $n \geqslant 3$.

(b) *Show that* $L(P_n) \simeq P_{n-1}$ *for any* $n \geqslant 2$.

(c) *For* $uv \in E$*, show that* $N_{L(G)}(uv) = \{uw : w \in N_G(u) \setminus \{v\}\} \sqcup \{wv : w \in N_G(v) \setminus \{u\}\}$.

(d) *For* $uv \in E$*, show that* $\deg_{L(G)} uv = \deg_G u + \deg_G v - 2$.

Theorem. *If G is a graph with at least one edge, then* $\chi'(G) \leqslant 2\Delta(G) - 1$.

Proof Exercise: $\chi'(G) = \chi(L(G))$

By Thm 10.9 (applied to $L(G)$) $\chi(L(G)) \leq \Delta(L(G)) + 1$. Then

$$\chi'(G) = \chi(L(G)) \leq 1 + \Delta(L(G))$$
$$= 1 + \max_{uv \in E(G)} \{ \deg_{L(G)} uv \}$$
$$= 1 + \max_{uv \in E(G)} \{ \deg_G u + \deg_G v - 2 \}$$
$$\leq 1 + 2\Delta(G) - 2 = 2\Delta(G) - 1.$$

$\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$

Next: Range for $\chi'(G)$ is smaller

Theorem (Vizing's Theorem, 10.12). *If G is any graph, then* $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$.

i.e: $\chi'(G) \to \Delta(G)$
$\to \Delta(G)+1$

Theorem. *For any integer* $n \geqslant 2$, *we have* $\chi'(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases}$

$\Delta(K_n) = n-1$

Proof:

$\chi'(K_n) \geq \begin{cases} n-1, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$

We know $\chi'(K_n) \geq \Delta(K_n) = n-1$ for any n

If n is odd, $\alpha'(K_n) = \frac{n-1}{2}$. Thus

$$\chi'(K_n) \geq \frac{|E(K_n)|}{\alpha'(K_n)} = \frac{\binom{n}{2}}{\frac{n-1}{2}} = \frac{\frac{n\cancel{(n-1)}}{\cancel{2}}}{\frac{\cancel{n-1}}{\cancel{2}}} = n$$

$\chi'(K_n) \leq \begin{cases} n-1, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$

Note: enough to show: $\chi'(K_n) \leq n-1$ for n even. We would have for an odd n, $\chi'(K_n) \leq \chi'(K_{n+1}) \leq n+1-1 = n$

Now we show $\chi'(K_n) \leq n-1 =: K$ for n even

don't need

Moreover, we will show

$$E(K_n) = M_0 \sqcup M_1 \sqcup \cdots \sqcup M_{K-1}$$ where each M_i is perfect matching

Label $V(K_n) = \{c\} \sqcup \{v_0, \ldots, v_{K-1}\}$

For each $0 \leq i \leq K-1$, define

$$M_i = \{cv_i\} \sqcup \{v_{i+x} v_{i-x} : x \in \{1, \ldots, \tfrac{K-1}{2}\}\}$$ Arithmetic mod K

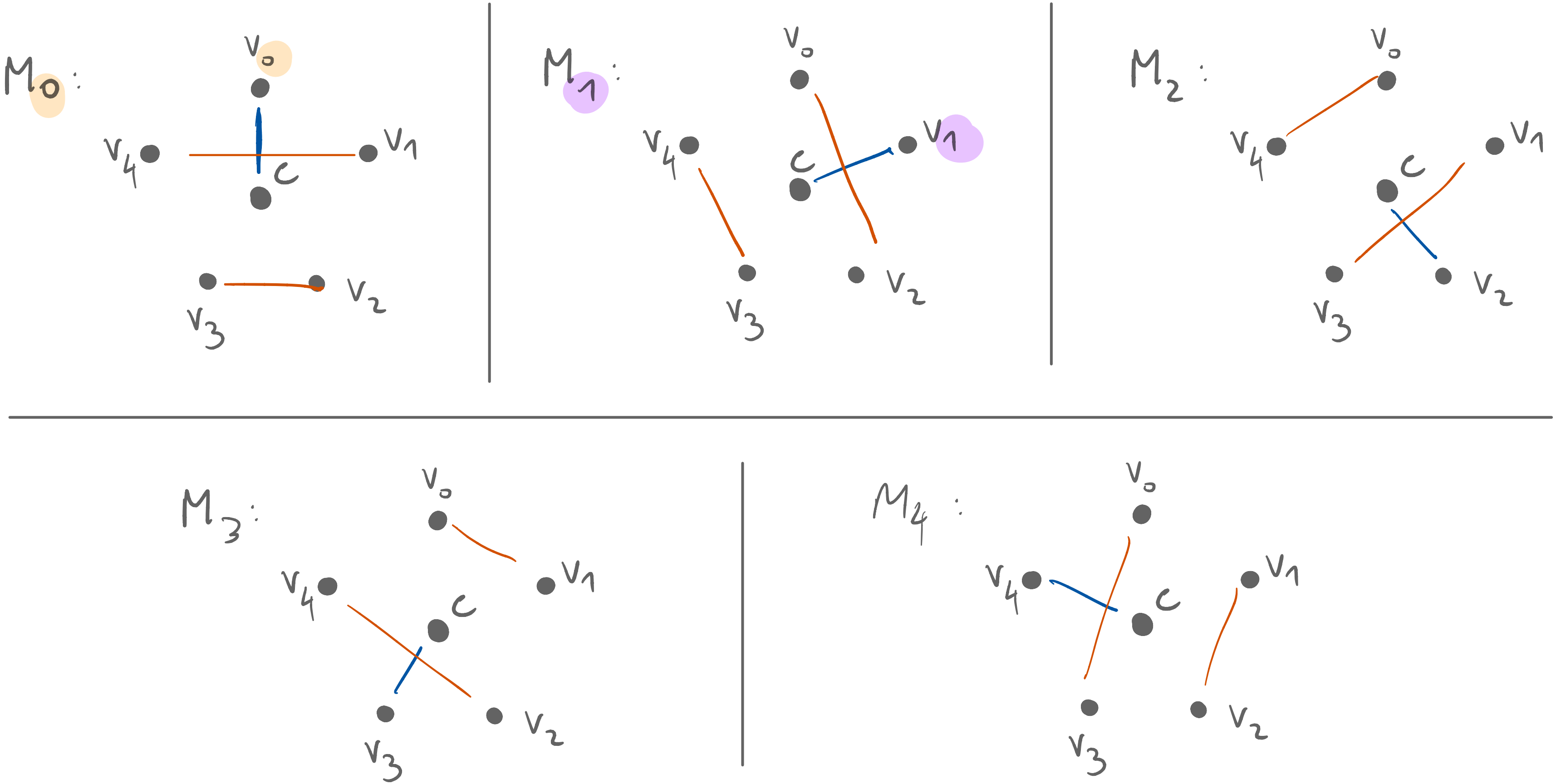
Idea: Form a regular K-gon with $v_0, \ldots, v_{K-1}$, put c in the center of the K-gon

For M_i add edge cv_i to M_i. Then, add all edges perpendicular to cv_i

Example: $n = 6$, $K = 5$

$$M_0 = \{cv_0\} \sqcup \{v_{0+1}v_{0-1}, v_{0+2}v_{0-2}\}$$
$$= \{cv_0\} \sqcup \{v_1v_4, v_2v_3\}$$

Example: $n = 6$



M0:
v0
v1
v2
v3
v4
c
M1:
v0
v1
v2
v3
v4
c
M2:
v0
v1
v2
v3
v4
c
M3:
v0
v1
v2
v3
v4
c
M4:
v0
v1
v2
v3
v4
c

Then each M_i is a perfect matching, and

$$E(K_n) = M_0 \sqcup \cdots \sqcup M_{k-1} \quad (k \text{ Terms})$$

Thus by prop, $\chi'(K_n) \leq k = n-1$ $\blacksquare$

Next:

Examples of $\chi'(G) = \Delta(G)$

$\chi'(G) = \Delta(G) + 1$

Tool: factorization.